The Islamia University Of Bahawalpur,
Department of Computer Science & IT
Bahawalnagar Campus
Course: Numerical Analysis **Program:** BSCS-V (Spring 2020)

Topic: Cubic Suplines method

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Polynomial Approximation:-

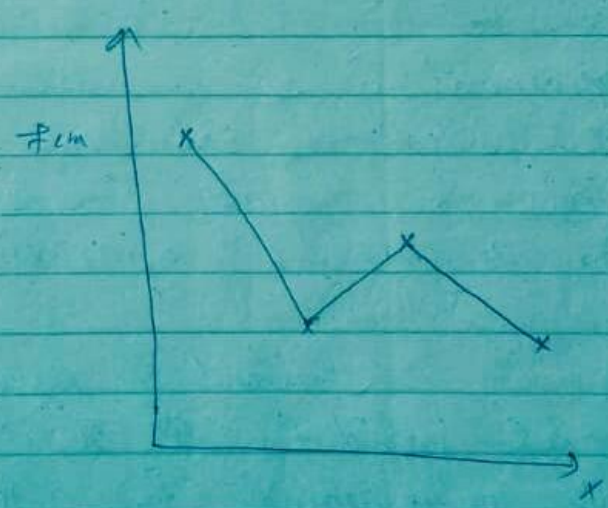
(iv)

⇒ By Splines (Cubic Splines) Method:-

We describing another approach for The polynomial approximation, i.e Splines (Mean clamping '[illegible]') or simply cubic splines method.

⇒ What are splines?

As we have studied earlier for given points.



The dot to dot interpolation There is a different straight lines b/w each pair of data points. These straight lines are known as linear splines.

⇒ Why are splines needed?

Fitting a polynomial to the data (using Lagrange polynomials, for example), works very well when there are a small number of data points.

But if there were 100 data points, it would be silly to try to fit a polynomial of degree '99' through all of them.
It would be a great deal work and anyway polynomials of high degree can be very oscillatory giving poor approximations b/w the data points to the underlying function.

⇒ How to apply Splines Method, for polynomial approximation:

Instead of using a polynomial valid for all x, we use one polynomial for $x_1 < x < x_2$, then a different polynomial for $x_2 < x < x_3$, then a different one again for $x_3 < x < x_4$, and so on i.e dot to dot interpolation; b/w the data points

The most commonly used splines are cubic splines.

→ We use a different polynomial of degree three b/w each pair of data points.

Let

$S = S(x)$ denote a cubic spline

Then

→ $S(x) = a_1(x-x_1)^3 + b_1(x-x_1)^2 + c_1(x-x_1) + d_1$

for interval

$(x_1 < x < x_2)$

→ $S(x) = a_2(x-x_2)^3 + b_2(x-x_2)^2 + c_2(x-x_2) + d_2$

For $(x_2 < x < x_3)$

→ $S(x) = a_3(x-x_3)^3 + b_3(x-x_3)^2 + c_3(x-x_3) + d_3$

For $(x_3 < x < x_4)$

$\vdots$ = - - - - - -

$\vdots$ = - - - - - -

$\vdots$ = - - - - - -

And we need to find $a_1, b_1, c_1, d_1, a_2, b_2, c_2, d_2$ - - - - - to determine the full form for the spline '$S(x)$'.

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Given the large number of quantities that have to be assigned (Four for every pair of adjacent data points) it is possible to give 'S' some very nice properties:

- $S(x_1) = f_1$, $S(x_2) = f_2$, $S(x_3) = f_3$, ... That 'S' interpolate the given data.
- $S'(x)$ is continuous at the data points. this means that there are no corners at the data points - the whole curve is smooth.
- $S''(x)$ is continuous. This reduces the occurrence of points of inflection appearing at the data points and leads to a smooth interpolant.

Even with all of these requirements, there are still two more properties we can assign to 'S'.

- A "natural cubic spline" is one for which $S''(x)$ is zero at the two end points.

The natural cubic spline is, in some sense, the smoothest possible spline for it minimise a measure of the curvature.

⇒ "How a natural cubic spline found?"

Suppose that there are 'N' data points. For a natural cubic spline we require

$$S''(x_1) = S''(x_N) = 0$$

And values of $S''(x)$ taken at the other data points are found from the cubic spline equation

$$\cancel{\begin{bmatrix} K_2 & h_2 & & \\ h_2 & K_3 & h_3 & \\ & \ddots & \ddots & \ddots \\ & & h_{N-3} & K_{N-2} \\ & & & h_{N-2} \end{bmatrix}}$$

$$\begin{bmatrix} K_2 & h_2 & & & \\ h_2 & K_3 & h_3 & & \\ & \ddots & \ddots & \ddots & \\ & & h_{N-3} & K_{N-2} & h_{N-2} \\ & & & h_{N-2} & K_{N-1} \end{bmatrix} \begin{bmatrix} S''(x_2) \\ S''(x_3) \\ \vdots \\ S''(x_{N-2}) \\ S''(x_{N-1}) \end{bmatrix} = \begin{bmatrix} r_2 \\ r_3 \\ \vdots \\ r_{N-2} \\ r_{N-1} \end{bmatrix}$$

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Here

$h_1 = x_2 - x_1$, $h_2 = x_3 - x_2$, $h_3 = x_4 - x_3$, $h_4 = x_5 - x_4$ ---

d

$k_2 = 2(h_1 + h_2)$

$k_3 = 2(h_2 + h_3)$

$k_4 = 2(h_3 + h_4)$

---

and

$$r_2 = 6\left(\frac{f_3 - f_2}{h_2} - \frac{f_2 - f_1}{h_1}\right)$$

$$r_3 = 6\left(\frac{f_4 - f_3}{h_3} - \frac{f_3 - f_2}{h_2}\right)$$

Admittedly the system of equations above looks unappealing, but this is a "nice" system of equation. The matrix above is of that type since the diagonal entry is always twice as big as the sum of off-diagonal enteries.

Once the system of equations is solved for the second derivatives S'', the spline 'S' can be found as follows.

$$a_i = \frac{S''(x_{i+1}) - S''(x_i)}{6h_i}$$

$$b_i = \frac{S''(x_i)}{2}$$

$$c_i = \frac{f_{i+1} - f_i}{h_i} - \left(\frac{S''(x_{i+1}) + 2S''(x_i)}{6} \right) h_i$$

$$d_i = f_i$$

We now present an example for illustrating this approach.

EXP No. 1

Find the natural cubic spline which interpolates the data

x_j	1	3	5	8
f_j	0.85	0.72	0.34	0.67

Solution $x_1 = 1, \quad x_2 = 3, \quad x_3 = 5, \quad x_4 = 8$

$f_1 = 0.85, \quad f_2 = 0.72; \quad f_3 = 0.34; \quad f_4 = 0.67$

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$h_1 = x_2 - x_1$
$= 3 - 1$
$h_1 = 2$

$h_2 = x_3 - x_2$
$= 5 - 3$
$h_2 = 2$

Similarly

$h_3 = x_4 - x_3$
$= 8 - 5$
$h_3 = 3$

For Natural Splines condition $S''(x_1) = S''(x_4) = 0$

For this case $N = 4$

So we have a Matrix is just 2×2 and pair of equations one

$$\underbrace{h_1 S''(x_1)}_{=0} + 2(h_1 + h_2) S''(x_2) + h_2 S''(x_3) = r_2$$

$$2(h_1 + h_2) S''(x_2) + h_2 S''(x_3) = 6\left(\frac{f_3 - f_2}{h_2} - \frac{f_2 - f_1}{h_1}\right)$$

$$2(2+2) S''(x_2) + 2 S''(x_3) = 6\left(\frac{0.34 - 0.72}{2} - \frac{0.72 - 0.85}{2}\right)$$

$$= 6\left(\frac{-0.38}{2} - \frac{0.13}{2}\right)$$

$$= 6(-0.125)$$

$$8 S''(x_2) + 2 S''(x_3) = -0.75 \quad \text{——(i)}$$

Similarly 2nd equation

$$h_2 S''(x_2) + 2(h_2 + h_3) S''(x_3) + \underbrace{h_3 S''(x_4)}_{=0} = r_3$$

$$r_3 = 6\left(\frac{f_4 - f_3}{h_3} - \frac{f_3 - f_2}{h_2}\right)$$

$$2 S''(x_2) + 10 S''(x_3) = 1.8 \quad \text{——(ii)}$$

Multiply by '4' we get

$$8 S''(x_2) + 40 S''(x_3) = 7.2 \quad \text{——(ii-A)}$$

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Now By solving the eq (i) & (ii)

Now by eq (i) & eq (ii-A)

$$8S''(x_2) + 40S''(x_3) = 7.2$$
$$\pm 8S''(x_2) \pm 2S''(x_3) = -0.75$$
$$38S''(x_3) = 7.95$$
$$\Rightarrow S''(x_3) = \frac{7.95}{38} = 0.2092105$$
$$\boxed{S''(x_3) = 0.209211}$$

Similarly By putting $S''(x_3) = 0.209211$ into eq (ii) we get

$$2S''(x_2) + 10(0.209211) = 1.8$$
$$2S''(x_2) + 2.09211 = 1.8$$
$$2S''(x_2) = 1.8 - 2.09211$$
$$2S''(x_2) = -0.29211$$
$$S''(x_2) = \frac{-0.29211}{2}$$
$$\boxed{S''(x_2) = -0.146053}$$

As our required Spline

$$S(x) = a_1(x-x_1)^3 + b_1(x-x_1)^2 + c_1(x-x_1) + d_1 \quad \text{(iii)}$$

Now find a_1, b_1, c_1 & d_1

As

$$a_i = \frac{S''(x_{i+1}) - S''(x_i)}{6h_i} \Rightarrow a_1 = \frac{S''(x_2) - S''(x_1)}{6h_1}$$
$$a_1 = \frac{-0.146053 - 0}{6 \times 2}$$

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$a_1 = -0.01217108$

Now $b_i = \frac{S''(x_i)}{2}$

$b_1 = \frac{S''(x_1)}{2} = \frac{0}{2} = 0$

$b_1 = 0$

$c_i = \frac{f_{i+1} - f_i}{h_i} - \left(\frac{S''(x_{i+1}) + 2S''(x_i)}{6}\right) h_i$

Similarly

$c_1 = -0.016316$

and

$d_i = f_i$

$d_1 = f_1 = 0.85$

Now our required spline is
putting above values i.e. (eq (iii))

$S(x) = -0.01217(x-1)^3 - 0.016316(x-1) + 0.85$

For interval $(1 < x < 3)$

Similarly for next interval $(3 < x < 5)$
$S(x) = a_2(x-x_2)^3 + b_2(x-x_2)^2 + c_2(x-x_2) + d_2$

$S(x) = 0.029605(x-3) - 0.073026(x-3)^2$

$- 0.162368(x-3) + 0.72$

~~For Interval $(3 < x < 5)$~~

And For interval $(5 < x < 8)$
$S(x) = a_3(x-x_3)^3 + b_3(x-x_3)^2 + c_3(x-x_3) + d_3$

$S(x) = -0.01162(x-5) + 0.104605(x-5)^2$

$- 0.099211(x-5) + 0.34$

For interval $(5 < x < 8)$

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Now for interval $(3 < x < 5)$

Spline is

$$S(x) = a_2(x - x_2)^3 + b_2(x - x_2)^2 + c_2(x - x_2) + d_2$$

First we find a_2, b_2, c_2, d_2

As

$$a_i = \frac{S''(x_{i+1}) - S''(x_i)}{6h_i} \quad ; \quad b_i = \frac{S''(x_i)}{2}$$

$$c_i = \frac{f_{i+1} - f_i}{h_i} - \left(\frac{S''(x_{i+1}) + 2S''(x_i)}{6}\right) h_i$$

$$d_i = f_i$$

Now

$$a_2 = \frac{S''(x_3) - S''(x_2)}{6h_2}$$

As we have calculated

$$S''(x_3) = 0.209211$$

And $\quad S''(x_2) = -0.146053$

$$h_2 = 2$$

$$a_2 = \frac{0.209211 - (-0.146053)}{6 \times 2}$$

$$a_2 = \frac{0.355264}{12}$$

$$a_2 = 0.02960533$$

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$b_2 = ?$

$$b_2 = \frac{S''(x_2)}{2} = \frac{-0.146053}{2}$$

$$b_2 = -0.0730265$$

$c_2 = ?$

$$c_2 = \frac{f_3 - f_2}{h_2} - \left(\frac{S''(x_3) + 2S''(x_2)}{6}\right) h_2$$

$$c_2 = \frac{0.34 - 0.72}{2} - \left(\frac{0.209211 + 2(-0.146053)}{6}\right) 2$$

$$c_2 = \frac{-0.38}{2} - \frac{(0.209211 - 0.292106)2}{6}$$

$$= -0.19 - \left(\frac{-0.082895}{6}\right) 2$$

$$c_2 = -0.19 - \left(\frac{-0.082895}{3}\right)$$

$$c_2 = -0.19 + 0.02763167$$

$$c_2 = \cancel{-0.2176317} \; -0.1623683$$

Now $d_2 = f_2 = 0.72$

Our ~~Root~~ required spline for interval $(3 < x < 5)$

$$S(x) = 0.02960533(x-3)^3 - 0.0730265(x-3)^2 - 0.1623683(x-3) + 0.72$$

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Similarly spline for interval $(5<x<8)$

$$S(x) = a_3(x-x_3)^3 + b_3(x-x_3)^2 + c_3(x-x_3) + d_3$$

Find a_3, b_3, c_3, d_3

$$a_3 = \frac{\overset{\to 0}{S''(x_4)} - S''(x_3)}{6h_3}$$

$h_3 = x_4 - x_3$
$h_3 = 8 - 5$
$h_3 = 3$

$$a_3 = \frac{0 - 0.209211}{6 \times 3}$$

$$a_3 = \frac{-0.209211}{18}$$

$$a_3 = -0.01162283$$

And $b_3 = \dfrac{S''(x_3)}{2} = \dfrac{0.209211}{2} = 0.1046055$

$$c_3 = \frac{f_4 - f_3}{h_3} - \left(\frac{\overset{\to 0}{S''(x_4)} + 2S''(x_3)}{6}\right) h_3$$

$$= \frac{0.67 - 0.34}{3} - \left(\frac{2(0.209211)}{6}\right) \times 3$$

$$= \frac{0.33}{3} - 0.209211$$

$$= 0.11 - 0.209211$$

$c_3 = -0.099211$; $d_3 = f_3 = 0.34$

So our required spline for interval $(5<x<8)$

$$S(x) = -0.01162(x-5)^3 + 0.104605(x-5)^2 - 0.099211(x-5) + 0.34$$

Best of Luck